JOURNAL OF COMPUTATIONAL PHYSICS 31, 21–41 (1979)

Boundary-Distribution Solution of The Helmholtz Equation for a Region with Corners*†

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Received May 4, 1976; revised May 3, 1978

A technique is described for the solution of the Helmholtz equation together with associated boundary conditions based on a generalization of a method used for the solution of the Dirichlet problem of potential theory, in which a dipole distribution is introduced on the boundary of a region to generate the potential inside. In order that the boundary conditions be satisfied, the distribution must be found as the solution of an integral equation. If the boundary is smooth, the equation is of Fredholm type, but if it has a corner the equation is singular. The problem of a sharp corner is analyzed, and properties of the solution are developed using the theory of singular integral equations. Direct use of the technique can be made impossible in some cases by the presence of "partner problem" eigenvalues. A simple method for avoiding this difficulty is presented.

I. INTRODUCTION

The use of integral equations to find solutions of Laplace's equation or the Helmholtz equation satisfying various boundary conditions has a long history, dating back at least to the work of Fredholm on a "new" method of solution for the Dirichlet problem [1]. Although the properties of these integral equations have been of great theoretical interest throughout this century, the advent of the large, high-speed computers has made the method of great practical utility. In particular, for the solution of the Helmholtz equation,

$$(\nabla^2 + \kappa^2)\psi(\mathbf{r}) = 0, \tag{1}$$

the use of a boundary integral equation for obtaining ψ numerically has two decided advantages over the alternative finite difference (FDM) or finite element (FEM) methods in which an approximation for the Laplacian operator is introduced. In the first place, only points on the boundary of a region are needed to obtain the solution so that the dimensionality of the space of unknowns is reduced by one. This can be of substantial benefit in reducing computer storage requirements for a desired calcula-

* This work was supported by the U. S. Department of Energy.

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0021-9991/79/040021-21$02.00/0

tional accuracy. Secondly, for "exterior" problems, in which the region involved is not closed, an "outgoing wave condition" *is automatically achieved by the proper choice of kernel function in the integral equation.* This condition is difficult to impose accurately in the FDM or FEM approaches.

Three different (but related) boundary integral equation methods have been developed for solution of the Helmholtz equation: 1. the solution is expressed in terms of an auxiliary monopole ("charge") distribution on the boundary of the region; 2. the solution is expressed in terms of an auxiliary dipole distribution on the boundary; or, 3. the solution is expressed in terms of the solution and its normal derivative on the boundary by making use of the Helmholtz representation which arises from Green's theorem [2]. In each of these methods an integral equation for the unknown is obtained by considering the limit of the integral representation for $\psi(\mathbf{r})$ in which $\mathbf{r}$ approaches the boundary. These three methods have been presented together using a consistent notation in a fine paper by Kleinman and Roach [3], so that their similarities and their differences can easily be seen. It is found that the integral equations typically involve one of two kernels, either that for the free-space solution for an isolated monopole, or the normal derivative of that solution at a surface. If the former kernel occurs (and the boundary is smooth), one must solve a Fredholm equation of the first kind, while for the latter the equation is of the second kind. Since the solution of equations of the first kind presents some difficulties not found for those of the second kind [4], it has been traditional to choose methods leading to the latter. Recently, however, Jaswon [5] and Symm [6] have shown that the former equations can be treated satisfactorily, so either approach can be useful. In this paper we will use the dipole representation for the Dirichlet problem, but since the kernels needed for the various approaches are closely related, some of the conclusions which will be reached can be readily adapted for any of the other choices.

As has been noted above, the integral equation which is obtained for the dipole distribution has a kernel of the Fredholm type (completely continuous) if the boundary is smooth, but it is singular if sharp corners are present. In the last few years a number of applications of the method to acoustic and electromagnetic radiation problems has been made [7–12], but in these no detailed analysis of the complications arising from sharp corners was done. In Section II of this paper we give a reasonably complete analysis of the properties of the dipole distribution in the vicinity of a corner. The behavior of the solution of the Helmholtz equation near a corner has already been determined by Meixner [13] and others [14–17] using techniques based on the differential equation, but we here provide an alternative development based on a direct treatment of the singular boundary integral equation which we believe has an inherent interest of its own.

A significant difficulty exhibited by the boundary integral equations is that the homogeneous part of a particular equation may be identical to that for another problem. Thus, from Table I of Kleinman and Roach [3] one sees, for example, that the exterior Neumann problem expressed using a dipole distribution, and the interior Dirichlet problem using a Helmholtz representation have the same homogeneous parts. Thus, if one wishes to solve the exterior Neumann problem at an eigenvalue of the interior Dirichlet problem a direct approach will be impossible. This difficulty is well-known [2, 18, 19]. If the Helmholtz representation is used, one finds that the inhomogeneous term in the integral equation is orthogonal to the eigensolution of the transposed homogeneous equation so that, by the well-known Fredholm theorems [20] a solution exists but it is not unique. It has been shown that, for the Helmholtz representation, if the second of the two integral equations is added to the system, one will then have a unique result [3, 19, 21, 22]. In Section III of this paper, we provide a simple alternative method of calculation for use when a dipole distribution is employed. This involves modifying the kernel so that its resolvent is not singular at the eigenvalue.

The ideas of this paper have been implemented in the development of a computer program for the solution of the Helmholtz equation in two dimensions for regions bounded by polygons. In a subsequent paper the numerical techniques used and some representative results are described.

II. The Dipole Distribution Near a Corner

A solution of the Helmholtz equation, Eq. (1), in a region V can be expressed in terms of a dipole distribution, $D(\mathbf{r})$, on the boundary of the region, S_V. Specifically, one can write:

$$\psi(\mathbf{r}) = \int_{S_V} D(\mathbf{r}')[\nabla' G(\mathbf{r}, \mathbf{r}')] \cdot d\boldsymbol{\sigma}',$$

where $G(\mathbf{r}, \mathbf{r}')$ is a solution of the equation

$$(\nabla_r{}^2 + \kappa^2) G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$

and the integral is taken over the surface, S_V, of the region [2]. If the region V is finite, only the singularity in G at $\mathbf{r} \to \mathbf{r}'$ is important, but if the region is unbounded then a further condition for $\mathbf{r} \to \infty$ must be imposed. In such cases, one is usually interested in the solution of wave-scattering problems, and one writes

$$\psi(\mathbf{r}) = \psi_{inc}(\mathbf{r}) + \psi_{sc}(\mathbf{r}).$$

Here, $\psi_{inc}(\mathbf{r})$ is a specified incident wave, and $\psi_{sc}(\mathbf{r})$ is the scattered wave. The latter is required to have only "outgoing" parts. In this paper we shall be interested in two-dimensional phenomena; in this case [23]

$$G(\mathbf{r}, \mathbf{r}') = -iH_0^{(1)}(\kappa \mid \mathbf{r} - \mathbf{r}' \mid)/4, \tag{2}$$

where $H_0^{(1)}$ is the Hankel function of the first kind. In two dimensions the volume, V, becomes an area in a plane, and the integral is taken over the bounding contour with

$d\sigma = -\hat{e}_z \times d\ell$, where $\hat{e}_z$ is a unit vector out of the plane and $d\ell$ is a line element along the contour. Using $H_0^{(1)\prime} = -H_1^{(1)}$, we can write

$$\psi_{sc}(\mathbf{r}) = \frac{i\kappa}{4} \oint_{S_V} D(\mathbf{r}') \frac{H_1^{(1)}(\kappa \mid \mathbf{r}' - \mathbf{r} \mid)}{\mid \mathbf{r}' - \mathbf{r} \mid} (\mathbf{r}' - \mathbf{r}) \cdot d\sigma'. \tag{3}$$

If one has a bounded region, then the total solution, $\psi(\mathbf{r})$, will typically be represented as in Eq. (3), but the (complex) Hankel function factor $iH_1^{(1)}$ can be replaced by the (real) Neumann function $(-N_1)$. In the Dirichlet problem, ψ is specified on the boundary, and one obtains the boundary integral equation by taking the limit as $\mathbf{r}$ approaches the boundary from within the region V. One finds:

$$f(\mathbf{r}) = \frac{D(\mathbf{r})}{2} + \frac{i\kappa}{4} \oint_{S_V} D(\mathbf{r}') \frac{H_1^{(1)}(\kappa \mid \mathbf{r}' - \mathbf{r} \mid)}{\mid \mathbf{r}' - \mathbf{r} \mid} (\mathbf{r}' - \mathbf{r}) \cdot d\sigma', \tag{4}$$

where $f(\mathbf{r})$ is the specified boundary value of $\psi(\mathbf{r})$. We propose to investigate the properties of this integral equation.

Although the boundary distribution technique can be applied directly to cases in which the boundary is smooth, i.e., satisfies a Liapunov condition [24], some additional analysis must be given if the boundary has sharp corners. In the former case, the kernel of the equation can be shown to be completely continuous and so the usual Fredholm theorems apply. On the other hand, if there are corners the kernel is singular.

To deal with this situation, we will consider a corner in a boundary and for simplicity we will assume that the two sides of the corner are straight. The angle between these two sides will be called α. Further, since the singular nature of the equation comes about because of the small-distance behavior of the kernel, we divide the kernel into a leading term which includes the most singular part, and a remainder which is completely continuous. Thus we write:

$$H_1^{(1)}(x) \equiv -\frac{2i}{(\pi x)} + R(x), \tag{5}$$

and we will focus attention principally on the first term.

If Eq. (5) is now introduced into Eq. (4), we find:

$$\begin{aligned} f(\mathbf{r}) = \frac{D(\mathbf{r})}{2} &+ \oint \frac{D(\mathbf{r}')(\mathbf{r}' - \mathbf{r}) \cdot d\sigma'}{2\pi \mid \mathbf{r}' - \mathbf{r} \mid^2} \\ &+ \frac{i\kappa}{4} \oint D(\mathbf{r}') \frac{R(\kappa \mid \mathbf{r}' - \mathbf{r} \mid)}{\mid \mathbf{r}' - \mathbf{r} \mid} (\mathbf{r}' - \mathbf{r}) \cdot d\sigma'. \end{aligned}$$

Let us now introduce the notation that $D_1(s)$ is $D(\mathbf{r})$ on side 1 of the corner where s is the distance from the corner, and $D_2(s)$ is $D(\mathbf{r})$ on side 2. With this notation, the equation can be explicitly written for $\mathbf{r}$ on side 1 as:

$$f_1(s) = \frac{D_1(s)}{2} + \frac{s \sin \alpha}{2\pi} \int_0^{\ell_2} \frac{D_2(s')\, ds'}{(s'^2 - 2s's \cos \alpha + s^2)}$$
$$+ \frac{i\kappa}{4} s \sin \alpha \int_0^{\ell_2} D_2(s') \frac{R[\kappa(s'^2 - 2s's \cos \alpha + s^2)^{1/2}]}{(s'^2 - 2s's \cos \alpha + s^2)^{1/2}}\, ds'$$
$$+ \frac{i\kappa}{4} \int_{B'} D(\mathbf{r}') \frac{H_1^{(1)}(\kappa \mid \mathbf{r}' - \mathbf{r} \mid)}{\mid \mathbf{r}' - \mathbf{r} \mid} (\mathbf{r}' - \mathbf{r}) \cdot d\sigma'. \tag{6}$$

where $f_1(s)$ is the boundary value of $f(\mathbf{r})$ on side 1. For a straight side there is no contribution from the distribution $D_1(s)$ to the potential on that side except for the term $D_1(s)/2$, because the vector $\mathbf{r}' - \mathbf{r}$ is perpendicular to the surface element. The length of side 2 is ℓ_2. The integral over B' is the contribution from the distribution other than the part on sides 1 and 2. This last integral is analytic as a function of s, since it is a finite integral and $\mid \mathbf{r}' - \mathbf{r} \mid > 0$ for $\mathbf{r}'$ on B' and $\mathbf{r}$ on side 1.

Similarly, for side 2 we have:

$$f_2(s) = \frac{D_2(s)}{2} + \frac{s \sin \alpha}{2\pi} \int_0^{\ell_1} \frac{D_1(s')\, ds'}{(s'^2 - 2s's \cos \alpha + s^2)} + \cdots,$$

where the $\cdots$ indicates terms similar to the R, B' terms for f_1.

To analyze the corner singularity, we introduce $D_\pm(s) = D_1(s) \pm D_2(s)$. We then obtain

$$\frac{D_\pm(s)}{2} \pm \frac{s \sin \alpha}{2\pi} \int_0^{\ell_m} \frac{D_\pm(s')\, ds'}{(s'^2 - 2s's \cos \alpha + s^2)} = F_\pm(s),$$

where ℓ_m is the lesser of ℓ_1 and ℓ_2, and $F_\pm(s)$ includes the contributions of $f_i(s)$ and the remainder of the equation coming from R, B', and the integral for the larger ℓ_i beyond ℓ_m. Obviously, these integral equations have a singular kernel as $s, s' \to 0$, and so some care must be used in dealing with them, either for analytic or numerical purposes.

We now make a Mellin transformation of the equations to obtain

$$\frac{\Delta_\pm(\xi)}{2} \pm \frac{\sin \alpha}{(2\pi)^2\, i} \int_{c-i\infty}^{c+i\infty} d\xi'\, \Delta_\pm(\xi') \int_0^\infty s^\xi\, ds \int_0^{\ell_m} \frac{(s')^{-\xi'}\, ds'}{s'^2 - 2s's \cos \alpha + s^2} = \Phi_\pm(\xi). \tag{7}$$

In this equation, $\Delta(\xi) \equiv \int_0^\infty D(s)\, s^{\xi-1}\, ds$. In obtaining Eq. (7), we have made the direct Mellin integration and have used the inverse relation:

$$D(s) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Delta(\xi)\, s^{-\xi}\, ds.$$

The choice of the constant c will be discussed later. The transform of the function $F_\pm(s)$ is $\Phi_\pm(\xi)$. In arriving at this equation we have interchanged the order of integration over ξ' with those over s', s, which can be justified a posteriori. Next we can evaluate the integrals over s' and s. It would be convenient at this point if one could interchange these integrations, but in fact the results can easily be shown to depend on the order chosen. The integral over s' can be evaluated by noting that $(s'^2 - 2s's \cdot \cos\alpha + s^2) = [s' - se^{i\alpha}][s' - se^{-i\alpha}]$, and then separating the factors using partial fractions; the resulting integral can be expressed in terms of hypergeometric functions [25]:

$$\int_0^{\ell_m} \frac{(s')^{-\xi'}\,ds'}{(s'^2 - 2s's\cos\alpha + s^2)} = -\frac{\ell_m^{1-\xi'}}{2i(1-\xi')\,s^2\sin\alpha}$$
$$\times\left[e^{-i\alpha}\,{}_2F_1\left(1, 1-\xi'; 2-\xi'; \frac{\ell_m}{se^{i\alpha}}\right) - e^{i\alpha}\,{}_2F_1\left(1, 1-\xi'; 2-\xi'; \frac{\ell_m}{se^{-i\alpha}}\right)\right]. \quad (8)$$

If $s > l_m$, the hypergeometric functions are analytic and have convergent power series expansions, but if $s < \ell_m$ we need the analytic continuation of the functions as can be obtained using Kummer's relations [26]:

$$\int_0^{\ell_m} \frac{(s')^{-\xi'}\,ds'}{(s'^2 - 2s's\cos\alpha + s^2)}$$
$$= \frac{1}{s\sin\alpha}\left\{\pi s^{-\xi'}\frac{\sin[(\pi-\alpha)\,\xi']}{\sin\pi\xi'}\right.$$
$$\left. - \frac{\ell_m^{-\xi'}}{2i\xi'}\left[{}_2F_1\left(1, \xi'; 1+\xi'; \frac{se^{i\alpha}}{\ell_m}\right) - {}_2F_1\left(1, \xi'; 1+\xi'; \frac{se^{-i\alpha}}{\ell_m}\right)\right]\right\}. \quad (9)$$

These results can be used to evaluate the final integral over s, using Eq. (9) for the portion of the integral in which $0 \leqslant s \leqslant \ell_m$, and Eq. (8) for the remainder. For our purposes, the most important term which arises thereby is that which comes from the first term on the right-hand side of Eq. (9):

$$\frac{\pi\ell_m^{\xi-\xi'}}{(\xi-\xi')}\,\frac{\sin[(\pi-\alpha)\,\xi']}{\sin\pi\xi'} = \frac{\pi\ell_m^{\xi-\xi'}r(\xi')}{(\xi-\xi')}. \quad (10)$$

The feature of this term of particular interest is the pole at $\xi' \to \xi$. Then other terms can be integrated using the series representations for the hypergeometric functions, with the result that:

$$\int_0^\infty s^\xi\,ds\int_0^{\ell_m}\frac{(s')^{-\xi'}\,ds'}{s'^2 - 2s's\cos\alpha + s^2}$$
$$= \frac{\ell_m^{\xi-\xi'}}{\sin\alpha}\left\{\frac{\pi\sin[(\pi-\alpha)\,\xi']}{(\xi-\xi')\sin\ \pi\xi'} + \sum_{n=1}^\infty \sin n\alpha\left[\frac{1}{(n-\xi')(n-\xi)} - \frac{1}{(n+\xi')(n+\xi)}\right]\right\}. \quad (11)$$

As a function of ξ', it is easily found that the integral has poles at the positive integers, while as a function of ξ there are poles at all the integers with the exception of $\xi = 0$. Since the integrals will only be convergent if $\mathrm{Re}\{\xi'\} < 1$, $-1 < \mathrm{Re}\{\xi\} < 1$, and $\mathrm{Re}\{\xi - \xi'\} > 0$, we must initially restrict the range of these variables to satisfy the inequalities. If we let the point ξ approach the integration contour subject to the last of the conditions, we then obtain an integral equation for $\Delta(\xi)$, of the form:

$$\frac{\Delta_\pm(\xi)}{2} \pm \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \frac{(\ell_m)^{\xi-\xi'}\, r(\xi')\, \Delta_\pm(\xi')\, d\xi'}{(\xi-\xi')} = \Phi_\pm(\xi), \tag{12}$$

where we have absorbed the series of terms from Eq. (11) into Φ.

This equation is in standard singular integral equation form, and thus may be treated using known techniques [27]. We begin by considering the homogeneous equation, and introduce a function

$$H(\xi) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{r(\xi')(\ell_m)^{\xi-\xi'}\, \Delta^{(0)}(\xi')\, d\xi'}{(\xi'-\xi)}$$

where $\Delta^{(0)}$ is a solution of the homogeneous equation. Clearly $H(\xi)$ is an analytic function in the finite half-planes defined by $\mathrm{Re}(\xi) \gtrless c$, and it has a discontinuity in crossing the contour of integration. If we define the $H^{(\pm)}(\xi)$ to be the functions obtained from the integral in which $\mathrm{Re}(\xi) \gtrless c$, respectively, together with their analytic continuations, we then easily find that

$$r(\xi)\, \Delta^{(0)}(\xi) = H^{(-)}(\xi) - H^{(+)}(\xi),$$

and so

$$\tfrac{1}{2}[H_\pm^{(+)}(\xi) - H_\pm^{(-)}(\xi) \pm r(\xi)\, H_\pm^{(+)}(\xi)]/r(\xi) = 0. \tag{13}$$

This equation can be used to deduce the analytic structure of $\Delta_\pm^{(0)}(\xi)$.

We eventually wish to obtain the analytic structure of $D(s)$, which will require using the inverse transform on $\Delta(\xi)$. For the latter step, in the limit $s \to 0$, the contour in the inverse transform can be closed on the left, and so the behavior of $D(s)$ is determined by singularities on the left of the contour. In this region $H^{(-)}(\xi)$ is clearly analytic, and so we can solve for $H^{(+)}(\xi)$ in terms of $H^{(-)}(\xi)$ using Eq. (13) to analytically continue $H^{(+)}(\xi)$ to the left of the contour. Thus we find:

$$H_\pm^{(+)}(\xi) = (1 \pm r(\xi))^{-1}\, H_\pm^{(-)}(\xi).$$

A solution of this equation can be obtained by taking the logarithm of the equation and then noting that $\ln H(\xi)$ is a function with a given discontinuity on the contour. The solution of this problem (the "Hilbert problem") then can be written [28]

$$H_\pm(\xi) = \exp\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\ln[1 \pm r(\xi')]}{\xi'-\xi}\, d\xi'\right\}$$

assuming that the integral converges. We then see that $H_\pm^{(-)}(\xi)$ is analytic and nonzero on the left of the contour, and if we use Eq. (13) to analytically continue $H_\pm^{(+)}(\xi)$, it is evident that $H_\pm^{(+)}(\xi)$ will also be analytic unless

$$1 \pm r(\xi) = 0.$$

At such points, $H_{\pm}^{(+)}(\pm)$ will generally have poles. Thus $\Delta_{\pm}^{(0)}(\xi)$ also has poles at such points.

The solution of Eq. (12) may now be obtained by introducing

$$\mathscr{H}(\xi) \equiv \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{r(\xi')(\ell_m)^{\xi-\xi'} \Delta(\xi')\, d\xi'}{(\xi' - \xi)} .$$

Then it is easily seen that

$$r(\xi)\, \Delta_{\pm}(\xi) = \mathscr{H}_{\pm}^{(-)}(\xi) - \mathscr{H}_{\pm}^{(+)}(\xi)$$

so that

$$(1 \pm r(\xi))\, \mathscr{H}_{\pm}^{(+)}(\xi) - \mathscr{H}_{\pm}^{(-)}(\xi) = -2r(\xi)\, \Phi_{\pm}(\xi). \tag{14}$$

Using

$$(1 \pm r(\xi)) = H_{\pm}^{(-)}(\xi)/H_{\pm}^{(+)}(\xi),$$

this equation can be written:

$$\frac{\mathscr{H}_{\pm}^{(+)}(\xi)}{H_{\pm}^{(+)}(\xi)} - \frac{\mathscr{H}_{\pm}^{(-)}(\xi)}{H_{\pm}^{(-)}(\xi)} = - \frac{2r(\xi)\, \Phi_{\pm}(\xi)}{H_{\pm}^{(-)}(\xi)} .$$

Again we have a discontinuity equation to satisfy and we obtain as the formal solution:

$$\mathscr{H}_{\pm}(\xi) = - \frac{H_{\pm}(\xi)}{i\pi} \int_{c-i\infty}^{c+i\infty} \frac{r(\xi')\, \Phi_{\pm}(\xi')}{H_{\pm}^{(-)}(\xi')} \frac{d\xi'}{\xi' - \xi} .$$

Since $\mathscr{H}_{\pm}^{(-)}(\xi)$ is analytic on the left of the contour, if we use Eq. (14) to obtain the analytic continuation of $\mathscr{H}_{\pm}^{(+)}(\xi)$, we finally find that

$$\Delta_{\pm}(\xi) = \frac{\pm \mathscr{H}_{\pm}^{(-)}(\xi) + 2\Phi(\xi)}{1 \pm r(\xi)} . \tag{15}$$

Thus, we can generally expect poles in $\Delta(\xi)$ in the left half plane wherever $1 \pm r(\xi) = 0$ on the left of the contour.

To complete the discussion, it is necessary to specify the contour; i.e., to determine c. In the first place, from the restriction on $\mathrm{Re}(\xi)$, we require that $-1 < c < 1$. In addition, the preceding development will only give a meaningful expression for $H(\xi)$ if $\ln[1 \pm r(\xi)] \to 0$ as $|\mathrm{Im}\, \xi| \to \infty$. It is easily seen that $r(\xi) \sim \exp[(|\pi - \alpha| - \pi) \cdot |\mathrm{Im}\, \xi|]$ as $|\mathrm{Im}\, \xi| \to \infty$, so $r(\xi) \to 0$. Thus the logarithm will approach zero at ∞, unless it has an imaginary part of the form $i\pi n$. To guarantee that this does not happen, we can choose $c = 0$, since $r(\xi)$ is real and nonzero on the imaginary axis. Any other c satisfying the limit restriction is equally acceptable as long as the contour would not thereby be distorted from the imaginary axis by going past a zero of $1 \pm r(\xi)$, since in such a case the logarithm would acquire an imaginary part at ∞. (In principle the contour could pass *both* a zero and a pole in $1 \pm r(\xi)$ and still have a well-behaved integral as $|\operatorname{Im}\xi| \to \infty$. This condition cannot be achieved with the restriction on c, however; the poles in $r(\xi)$ are at $\xi = n$, $n \neq 0$.)

We now can conclude that $D(s)$ will behave as $\sim s^{-\xi_n}$ as $s \to 0$, where ξ_n is a pole in the transform, $\Delta(\xi)$. Such poles will appear if

$$\sin \pi\, \xi_n = \mp \sin(\pi - \alpha)\, \xi_n \tag{16}$$

if $\xi \neq 0$. In the case of Δ_+, the solutions of this equation are

$$\xi_n^{(+)} = -\frac{(2n-1)\,\pi}{\alpha},\ -\frac{2(n-1)\,\pi}{2\pi - \alpha} \tag{17A}$$

and in the case of Δ_-,

$$\xi_n^{(-)} = -\frac{2n\pi}{\alpha},\ -\frac{(2n-1)\,\pi}{2\pi - \alpha}, \tag{17B}$$

where n is any positive integer.

In addition to these poles, we must consider other possible singularities in $\Delta(\xi)$. Since $\mathscr{H}^{(-)}(\xi)$ is analytic, the only other possibility would be singularities in $\Phi(\xi)$. In fact, $\Phi(\xi)$ in part comes from contributions to $f(s)$ arising from distributions on the other boundaries, B', and since these contributions will be analytic near $s = 0$, this part of $\Phi_\pm(\xi)$ will be the transform of functions which have power series expansions; i.e., they have poles at the negative integers. If the explicit form of $r(\xi)$ is inserted into Eq. (15), however, we find

$$\Delta(\xi) = \frac{[\pm \mathscr{H}_\pm^{(-)}(\xi) + 2\Phi(\xi)] \sin \pi\xi}{[\sin \pi\xi \pm \sin(\pi - \alpha)\,\xi]},$$

so that any poles which appear in $\Phi(\xi)$ at the negative integers are cancelled by the factor $\sin \pi\xi$. On the other hand, if the boundary value $f_1(s)$ or $f_2(s)$ has a nonanalytic behavior as $s \to 0$ the singularities in $\Phi(\xi)$ generated thereby will not be cancelled.

Finally, we must consider singularities which are generated by parts of the kernel $H_1^{(1)}(\kappa r)/r$ other than the most singular one which has already been treated. We will see that if $\Delta(\xi)$ has a pole at $-\xi$, then additional poles will be generated at $-\xi - 2$, $-\xi - 4, \ldots$. Further, we will show that if ξ satisfies Eq. (15), then the contribution from these additional terms can be summed to provide a term in $D(s)$ proportional to the Bessel function $J_\xi(\kappa s)$. The proof consists of a demonstration that if $D(s) = J_\xi(\kappa s)$, then the integral operator of Eq. (4) carried over a side $0 \leqslant s' \leqslant \ell$ generates two terms: one proportional to $J_\xi(\kappa s)$ which is consistent with Eq. (4), and a second which is analytic. As has been seen, the latter makes a contribution to $\Phi(\xi)$ in Eq. (7) which produces no poles in $\Delta(\xi)$, and hence does not affect the analytic behavior of $D(s)$ as $s \to 0$.

For this proof we therefore consider

$$\int_0^\ell \frac{H_1^{(1)}(\kappa\, |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|}\, J_\xi(\kappa s')\, ds',$$

where $|\mathbf{r}-\mathbf{r}'| = [(s'-se^{i\alpha})(s'-se^{-i\alpha})]^{1/2}$. If the Hankel function is expressed using a Neumann and a Bessel function, the latter produces an integral which is an analytic function of s, since $J_1(z)/z$ is an entire function of z^2. Thus we only need consider

$$\nu(s, \alpha) \equiv \int_0^{\ell} \frac{N_1(\kappa\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} J_\xi(\kappa s')\, ds'.$$

The integrand has branch points in the complex s' plane at $s' = 0$, $se^{i\alpha}$, $se^{-i\alpha}$. If we introduce cuts in the plane as shown in Fig. 1, we can then write $\nu(s, \alpha)$ as an integral over the contour C:

$$\nu(s, \alpha) = (1 - e^{2\pi i\xi})^{-1} \int_C \frac{N_1(\kappa\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} J_\xi(\kappa s')\, ds'.$$

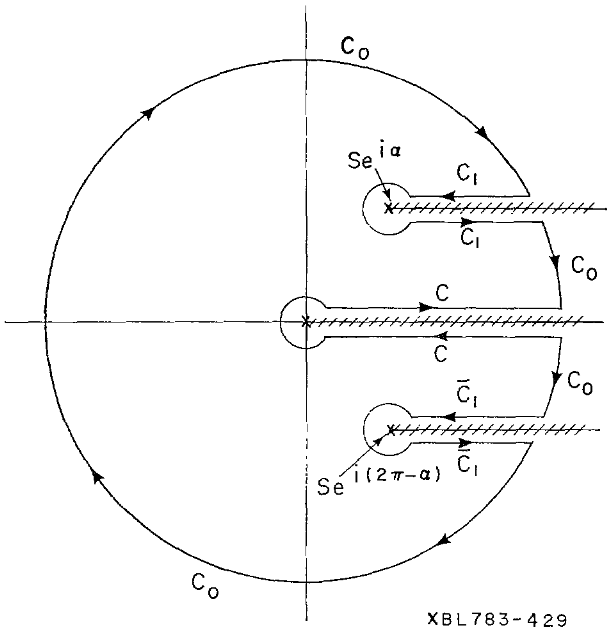

FIG. 1. Contour used to evaluate the integral of the Hankel function, $H_1^{(1)}$, times the Bessel function, J_ξ (see text).

Assuming that $s < \ell$, the contour may be distorted into three parts as shown: C_1 and $\bar{C}_1$ over the two cuts, and the parts of the circle C_0. The portion of $\nu(s, \alpha)$ contributed by C_0 gives an analytic function which we now ignore. Since

$$N_1(z) = -\frac{2}{\pi z} + \frac{2}{\pi} \log z \cdot J_1(z) + \psi_1(z),$$

where $\psi_1(z)$ is an entire function, odd in z, the integrals over C_1 and $\bar{C}_1$ include only a pole and a logarithmic discontinuity contribution. Thus one finds

$$\begin{aligned}\nu(s, \alpha) = (1 - e^{2\pi i \xi})^{-1} \Big\{ &- \frac{2}{\kappa s \sin \alpha} [J_\xi(\kappa s e^{i\alpha}) - J_\xi(\kappa s e^{i(2\pi - \alpha)})] \\ &+ 2i \int_{se^{i\alpha}}^{\ell} \frac{J_1(\kappa \mid \mathbf{r} - \mathbf{r}' \mid)}{\mid \mathbf{r} - \mathbf{r}' \mid} J_\xi(\kappa s') \, ds' \\ &+ 2i \int_{se^{i(2\pi-\alpha)}}^{\ell} \frac{J_1(\kappa \mid \mathbf{r} - \mathbf{r}' \mid)}{\mid \mathbf{r} - \mathbf{r}' \mid} J_\xi(\kappa s') \, ds' \Big\} + \text{A.F.}, \end{aligned} \tag{18}$$

where A.F. represents an analytic function of s. The phase of the argument of $J_\xi(\kappa s')$ has been fixed on the basis of the cut from $s' = 0$ to ∞ in Fig. 1. The integrals over s' can be rewritten as integrals over s' from the origin to $se^{i\alpha}$ or $se^{i(2\pi-\alpha)}$ plus an analytic term, so we wish to calculate

$$j(s, \alpha) \equiv \int_0^{se^{i\alpha}} \frac{J_1(\kappa \mid \mathbf{r} - \mathbf{r}' \mid)}{\mid \mathbf{r} - \mathbf{r}' \mid} J_\xi(\kappa s') \, ds'.$$

If we expand both Bessel functions in power series, we find

$$\begin{aligned} j(s, \alpha) = \left(\frac{\kappa}{2}\right)^{\xi+1} &\sum_{m=0}^{\infty} \frac{1}{m! \, (m+1)!} \sum_{n=0}^{\infty} \frac{(-\kappa^2/4)^{m+n}}{n! \, \Gamma(\xi + n + 1)} \\ &\times \int_0^{se^{i\alpha}} (s')^{\xi+2n} [(s' - se^{i\alpha})(s - se^{-i\alpha})]^m \, ds'. \end{aligned}$$

The integral can be evaluated easily after expanding the factor $(s' - se^{-i\alpha})^m$ using the binomial theorem. Thus:

$$\begin{aligned} j(s, \alpha) = \left(\frac{\kappa}{2}\right)^{\xi+1} &\sum_{m=0}^{\infty} \frac{(-\kappa^2 s^2/4)^m}{(m+1)!} \sum_{n=0}^{\infty} \left(-\frac{\kappa^2}{4}\right)^n \frac{(se^{i\alpha})^{\xi+2n+1}}{n! \, \Gamma(\xi + n + 1)} \\ &\times \sum_{k=0}^{m} \binom{m}{k} (-e^{2i\alpha})^k \frac{\Gamma(\xi + 2n + k + 1)}{\Gamma(\xi + 2n + m + k + 1)}. \end{aligned}$$

To carry out the sums, we change variables to $M = m + n$, and $\ell = k + n$, and then interchange the order of summation between ℓ and n to get:

$$\begin{aligned} j(s, \alpha) = \left(\frac{\kappa s e^{i\alpha}}{2}\right)^{\xi+1} &\sum_{M=0}^{\infty} \left(-\frac{\kappa^2 s^2}{4}\right)^M \sum_{\ell=0}^{M} \frac{(-e^{2i\alpha})^\ell}{(M-\ell)! \, \Gamma(\xi + M + \ell + 2)} \\ &\times \sum_{n=0}^{\ell} \frac{(-1)^n \, \Gamma(\xi + \ell + n + 1)}{(M - n + 1) \, n! \, (\ell - n)! \, \Gamma(\xi + n + 1)}. \end{aligned}$$

The sum over n can be done if we note that

$$\frac{\Gamma(\xi + \ell + n + 1)}{\Gamma(\xi + n + 1)} = x^{-\xi-n} \left(\frac{d}{dx}\right)^{\ell} x^{\xi+\ell+n}.$$

Then the sum, S, can be written as

$$S = \int_1^{\infty} dx\, x^{-\xi-M-2} \left(\frac{d}{dx}\right)^{\ell} [x^{\xi+\ell}(1 - x)^{\ell}].$$

The integral can easily be evaluated, giving

$$S = \frac{(-1)^{\ell}\,(M - \ell)!}{(M + 1)!}\, \frac{\Gamma(\xi + M + \ell + 2)}{\Gamma(\xi + M + 2)}.$$

Thus the sum over ℓ becomes:

$$\sum_{\ell=0}^{M} e^{2i\alpha} = e^{i\alpha M}\, \frac{\sin \alpha(M + 1)}{\sin \alpha}.$$

We therefore obtain

$$j(s, \alpha) = \frac{(\kappa s e^{i\alpha}/2)^{\xi+1}}{\sin \alpha} \sum_{M=0}^{\infty} \frac{(-\kappa^2 s^2/4)^M\, e^{i\alpha M} \sin \alpha(M + 1)}{(M + 1)!\, \Gamma(\xi + M + 2)}.$$

This term must be combined with one in which α is replaced by $(2\pi - \alpha)$:

$$j(s, \alpha) + j(s, 2\pi - \alpha) = \frac{(\kappa s/2)^{\xi+1}\, e^{i\pi\xi}}{\sin \alpha} \sum_{M=1}^{\infty} \frac{(-\kappa^2 s^2/4)^{M-1}}{M!\, \Gamma(\xi + M + 1)}$$
$$\times \{\sin[(\pi - \alpha)\, \xi] - \sin[(\pi - \alpha)\, \xi - 2\alpha M]\}. \quad (19)$$

To complete the calculation of $\nu(s, \alpha)$, we expand

$$J_{\xi}(\kappa s e^{i\alpha}) - J_{\xi}(\kappa s e^{i(2\pi-\alpha)})$$
$$= -2i \left(\frac{\kappa s}{2}\right)^{\xi} e^{i\pi\xi} \sum_{M=0}^{\infty} \frac{(-\kappa^2 s^2/4)^M}{M!\, \Gamma(\xi + M + 1)} \sin[(\pi - \alpha)\, \xi - 2\alpha M]. \quad (20)$$

If the results of Eqs. (19) and (20) are combined with Eq. (18), we then find:

$$\nu(s, \alpha) = -\frac{2 \sin[(\pi - \alpha)\, \xi]}{\kappa s \sin \alpha \sin \pi\xi}\, J_{\xi}(\kappa s) + \text{A.F.} \quad (21)$$

Finally, the kernel in the integral equation for a point on a side of the corner is $-\kappa s \sin \alpha \cdot N_1(\kappa \mid \mathbf{r}' - \mathbf{r} \mid)/(4 \mid \mathbf{r}' - \mathbf{r} \mid)$, so that if ξ satisfies Eq. (17) then the series of terms of the form $s^{\xi+2M}$ can be summed to give $J_{\xi}(\kappa s)$. Thus, unless the boundary conditions introduce other singularities for nonintegral ξ's there are no other singularities and hence we can expand $D(s)$ in an infinite series of such Bessel functions. It may be noted that if pole singularities in $\Delta(\xi)$ are introduced by the boundary conditions at values of ξ which do not satisfy Eq. (17), then one still would have a series of terms for $D(s)$ of the form $s^{\xi+2M}$, but the coefficients would no longer be related to the Bessel series. Lastly, we mention that although the analytic parts of $\nu(s, \alpha)$ and other analytic contributions to the integral equation do not influence the analytic contributions to the integral equation do not influence the analytic form of $D(s)$, it is clear that they play an essential role in determining the *residue* of the poles in $\Delta(\xi)$ at the values of Δ given by Eq. (17).

A few comments are appropriate at this point: In deducing the analytic form of the solution, we have assumed that the unknown functions on the remainder of the boundary away from the corner of interest can be treated as if they were known. The legitimacy of this approach can be rigorously established using the Carlemann-Vekua method for solving singular equations [29]. In this technique one uses the solution of the "dominant part" of the equation as illustrated here in solving Eq. (12) to convert the singular kernel to one which is only weakly singular and hence can be solved using Fredholm theory. We did not feel that such an approach, which does little else than to increase the complexity of notation and the bulk of the equations, was particularly illuminating and so we have chosen the more heuristic approach given above. We refer the interested reader to the rigorous treatment of singular integral equations for a complete discussion.

In addition to the behavior just deduced, in certain particular cases it is possible to obtain a different type of function in $D(s)$. This will occur if the denominator in Eq. (15) has a double zero at $\xi = \xi_0$; i.e., an "inside" and an "outside" ξ as given by Eq. (17) are identical. This will only occur in the even solution if

$$\alpha = \frac{(2n-1)}{(2m+2n-3)} 2\pi,$$

and in the odd case if

$$\alpha = \frac{2n}{(2m+2n-1)} 2\pi,$$

where m, n are positive integers. In such a case $\Delta(\xi)$ will have a double pole at ξ_0. Since such a double pole can be obtained by the coalescence of two single poles, it is strongly suggested that we can expect a solution of $D(s)$ of the form $[\partial J_\xi(\kappa s)/\partial\xi]_{\xi=\xi_0}$. The proof of this conjecture is quite simple.

It has already been demonstrated that

$$\frac{1}{2}\left[J_\xi(\kappa s) \pm \frac{i\kappa s \sin\alpha}{4} \int_0^{\ell} \frac{H_1^{(1)}(\kappa \mid \mathbf{r}-\mathbf{r}' \mid)}{\mid \mathbf{r}-\mathbf{r}' \mid} J_\xi(\kappa s')\, ds'\right]$$
$$= \frac{1}{2}\left\{1 \pm \frac{\sin[(\pi-\alpha)\,\xi]}{\sin \pi\xi}\right\} J_\xi(\kappa s) + \text{analytic terms.} \tag{22}$$

Since this relation is valid for values of ξ in the neighborhood of ξ_0, it can be differentiated on ξ. However, if $\sin \pi\xi \pm \sin[(\pi-\alpha)\xi]$ has a double zero $\xi = \xi_0$, its first derivative also vanishes there. Thus the nonanalytic part of the integral equation which depends on either $J_\xi l(\kappa s)$ or $\partial J_\xi l(\kappa s)/\partial \xi_0$ cancels out of the equation, and such terms are therefore allowed in the expansion of $D(s)$.

Finally, we consider the relation between the behavior of $D(s)$ near a corner and that of $\psi(\mathbf{r})$, where $\mathbf{r}$ is not on the boundary. Here:

$$\psi(\mathbf{r}) = \frac{i\kappa}{4} \oint \frac{H_1^{(1)}(\kappa \mid \mathbf{r} - \mathbf{r}' \mid)}{\mid \mathbf{r} - \mathbf{r}' \mid} D(\mathbf{r}')(\mathbf{r}' - \mathbf{r}) \cdot d\sigma'.$$

If we represent the point $\mathbf{r}$ in polar coordinates (r, θ) we replace Eq. (22) with

$$\psi(\mathbf{r}) = \frac{1}{2} \left\{ \frac{\sin[(\pi - \theta)\, \xi] \pm \sin[(\pi - \alpha + \theta)\, \xi]}{\sin \pi\xi} \right\} J_\xi(\kappa r) + \text{A.F.}$$

The trigonometric terms in the numerator of the brackets can be combined to give

$$\begin{aligned} &\sin[(\pi - \theta)\, \xi] \pm \sin[(\pi - \alpha + \theta)\, \xi] \\ &\quad = 2 \begin{cases} \sin\left[\dfrac{(2\pi - \alpha)\, \xi}{2}\right] \cos\left[\dfrac{(\alpha - 2\theta)\, \xi}{2}\right] & \text{(even)} \\ \cos\left[\dfrac{(2\pi - \alpha)\, \xi}{2}\right] \sin\left[\dfrac{(\alpha - 2\theta)\, \xi}{2}\right] & \text{(odd).} \end{cases} \end{aligned}$$

These functions are zero at the boundaries, $\theta = 0$ and $\theta = \alpha$, for ξ's which satisfy Eq. (17), but if ξ satisfies the "exterior" condition (which depends on $(2\pi - \alpha)$), then the solution ψ has no term dependent on the corresponding $J_\xi(\kappa r)$ anywhere inside the corner, even though $D(s)$ has such a term. [We note that the ξ's are determined by the homogeneous part of the integral equation, and thus relate to solutions which vanish on the boundary. The specified boundary value for ψ enters the equation as the inhomogeneous term in the integral equation.]

The results for the corner behavior deduced above can be compared with previous analyses based on direct treatment of the Helmholtz equation. The exact solution for the diffraction of scalar or electromagnetic waves by a wedge gives terms of the form $s^{(n\pi/\alpha)+2k}$ near the edge of the wedge, where n, k are integers [15, 16]. Maue based his analysis of diffraction on the Helmholtz representation [14], but he obtained the behavior of the solution near a corner by considering the partial differential equation there, rather than directly using the integral equation as we have done. A quite general analysis of the solution of elliptic partial differential equations together with associated boundary conditions has been carried out by Wigley [17]. Finally, it should be mentioned that Meixner [13] used the physically reasonable condition that the field energy density in the electromagnetic case should be integrable in order to exclude terms in the solution which are too singular as $s \to 0$. The results obtained here completely agree with these various treatments since the first set of ξ's in Eqs. (17A, B) match the behavior obtained for solutions of the partial differential equation, while the second set, which depends on $(2\pi - \alpha)$, does not produce a $J_\xi(\kappa r)$ behavior in $\psi(r, \theta)$. It may be noted that the behavior of the solution to the more general problem discussed by Wigley can have certain logarithmic terms which do not occur in the present case. The logarithmic terms which would appear here in $D(s)$ in the case of a double pole in $\Delta(\xi)$, as discussed above, can easily be seen to vanish in $\psi(\mathbf{r})$.

Although we have demonstrated that the results obtained here for the dipole distribution are consistent with earlier work, some doubt may still remain regarding their equivalence. Clearly the analysis used here depends only on the form of the kernel in the integral equation, and, as shown by Kleinman and Roach [3], for any particular boundary value problem the same kernel also occurs in a related Helmholtz-representation integral equation for another problem. For example, the interior Dirichlet problem using the dipole representation (IDD) has the same kernel as does the exterior Neumann problem using the Helmholtz representation (ENH). In terms of the solution to the equation, however, there is a difference: in the Helmholtz representation the solution must have the properties deduced using the partial differential equation, while the dipole distribution is not so restricted since it is only an auxiliary function. To gain some insight into these differences, we can use an argument given by Lamb [2] to relate the dipole distribution to a superposition of two solutions for the Helmholtz representation.

We consider the Helmholtz representation for the interior of a region, V, bounded by S_V. As is well known, one has

$$\oint_{S_V} \{G_n(\mathbf{r}, \mathbf{r}')\, \psi(\mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\, \psi_n(\mathbf{r}')\}\, ds' = \begin{cases} \psi(\mathbf{r}) & \mathbf{r} \in V \\ 0 & \mathbf{r} \notin V \end{cases}$$

where $G(\mathbf{r}, \mathbf{r}')$ is the usual free-space solution of the Helmholtz equation as given in Eq. (2) (for two dimensions), and n indicates the normal derivative taken with respect to $\mathbf{r}'$ out of V. (We again note that if the domain in $\mathbf{r}$ for ψ or ψ' is unbounded, for the equation to be true it is necessary that the solution for that region must satisfy an asymptotic outgoing wave condition, while for the bounded region the solution must remain finite throughout. These two conditions lead to different solutions since one cannot have a solution in all space which satisfies both.) If we use this relation once for the desired solution ψ defined *inside* V and again for an auxiliary ψ' defined *outside* V, on adding the two representations, we obtain

$$\psi(\mathbf{r}) = \int_{S_V} \{G_n(\mathbf{r}, \mathbf{r}')[(\psi(\mathbf{r}') + \psi'(\mathbf{r}')] - G(\mathbf{r}, \mathbf{r}')[\psi_n(\mathbf{r}') + \psi_n'(\mathbf{r}')]\}\, ds', \qquad (23)$$

where $\mathbf{r} \in V$. Thus, if we choose $\psi_n'(\mathbf{r}) = -\psi_n(\mathbf{r})$ on S_V we obtain the dipole representation for the solution as $D(\mathbf{r}) \equiv \psi(\mathbf{r}) + \psi'(\mathbf{r})$. From this one sees that since ψ must satisfy the edge conditions for the internal solution, and ψ' must satisfy the external conditions, the dipole distribution will include both, as exhibited by Eq. (17).

It may be helpful to indicate one further difference between the dipole and the Helmholtz representations even when the kernels are identical. For example, the IDD case satisfies

$$\frac{D(\mathbf{r})}{2} + \oint_{S_V} G_n(\mathbf{r}, \mathbf{r}')\, D(\mathbf{r}')\, ds' = \psi(\mathbf{r}),$$

where $\psi(\mathbf{r})$ is the (given) boundary value for the solution, while the ENH case has the equation

[illegible]

where $\psi_n(\mathbf{r})$ is the (given) normal derivative of the solution on the boundary. Clearly, in the latter case the solution ψ must be consistent with ψ_n, and the lack of internal-type ξ's in ψ must result from the specific form for the inhomogeneous part which of course differs from that of the dipole case. It seems desirable to demonstrate this result by a direct proof based on the integral equation.

Finally we note that, as shown by Kleinman and Roach [3], the kernel in the Helmholtz representation for a problem is the transpose of that arising in the dipole representation for the same case. This results in the replacement of s by s' in Eq. (6) for f_1 and makes a corresponding change in the equation for f_2. Otherwise the analysis for $D(s)$ can be carried out in the same way as before, and it is easily seen that the resulting behavior for the solution has each $-\xi$ shifted to $-\xi - 1$. In this case, however, the unknown function obtained is the normal derivative, ψ_n, so that the result is to be expected.

III. Elimination of Difficulties Caused by Eigenvalues of the Dipole Integral Equation

As has been stated earlier, the dipole representation suffers from difficulties arising from spurious eigenvalues associated with its "partner" problem [2]. This condition exists as well for the Helmholtz representation but in that case it can be shown that the inhomogeneous term in the integral equation is orthogonal to the eigensolution of the transposed kernel, so according to the Fredholm alternatives a solution exists although it will not be unique. By use of an additional condition [3, 19, 21, 22], however, the lack of uniqueness can be removed.

In this section we wish to demonstrate that the dipole representation can be simply modified so that the difficulty with such unwanted solutions is avoided. This technique is similar to the Schmidt method for solving Fredholm equations [30], but has a different motivation. Let us write the dipole integral equation symbolically as

$$\frac{D(s)}{2} + \int_0^L K(s, s')\, D(s')\, ds' = \Phi(s), \tag{24}$$

where L is the total path length around the boundary. We are now primarily interested in obtaining a solution to the integral equation even if κ is close to an eigenvalue for the partner problem. We first make the substitution

$$D(s) = A\, D_0(s) + D_1(s), \tag{25}$$

where $D_0(s)$ is the eigensolution of the homogeneous part of the integral equation, Eq. (24), and A is to be determined. We also make the further substitution

$$K(s, s') \equiv k(s, s') + f(s)g(s'), \tag{26}$$

where f, g are to be specified. In numerical calculations, we have chosen $g(s) = 1$ and $f(s) = -D_0(s)/2$ with excellent results. With these substitutions, we then obtain a modified integral equation:

$$\begin{aligned}\frac{D_1(s)}{2} &+ \int_0^L k(s, s')\, D_1(s')\, ds' + f(s) \int_0^L D_1(s')\, g(s')\, ds' \\ &= \Phi(s) - A \left\{\int_0^L K(s, s')\, D_0(s')\, ds' + D_0(s)/2\right\}.\end{aligned} \tag{27}$$

It is clear that if $\int_0^L D_1(s')g(s')\, ds' = 0$, the kernel $K(s, s')$ can be replaced by $k(s, s')$ in the equation, and one still has a valid solution of Eq. (24). Since the equation for D_1 is linear, the condition is easily achieved. We must only solve the equation

$$\frac{\chi(s)}{2} + \int_0^L k(s, s')\, \chi(s')\, ds' = \zeta(s)$$

twice: once for $\zeta(s) = \Phi(s)$ to get $\chi(s) = D_1^{(\Phi)}(s)$, and again for $\zeta(s) = D_0(s)/2 + \int_0^L K(s, s')\, D_0(s')\, ds'$ to get $D_1^{(0)}(s)$. (Note that in this case, we can write

$$\zeta(s) = \int_0^L [K(s, s') - K_0(s, s')]\, D_0(s')\, ds',$$

where K_0 is the kernel for $\kappa = \kappa_0$.) Then we have

$$D(s) = AD_0(s) + D_1^{(\Phi)}(s) + AD_1^{(0)}(s),$$

and A must be chosen so that

$$A = -\int_0^L g(s)\, D_1^{(\Phi)}(s)\, ds \Big/ \int_0^L g(s)\, D_1^{(0)}(s)\, ds. \tag{28}$$

The modified kernel, k, will not have the eigenvalue κ_0 in its spectrum, and so approximate calculational techniques will normally have well-conditioned matrices. Further, the singular nature of the solution as $\kappa \to \kappa_0$ is made explicit. In this limit, we see that the $\zeta(s)$ which generates $D_1^{(0)}(s)$ becomes

$$\zeta(s) \approx (\kappa - \kappa_0) \int_0^L \left(\frac{\partial K(s, s')}{\partial \kappa}\right)_{\kappa=\kappa_0} D_0(s')\, ds',$$

so that $A(\kappa - \kappa_0)$ will tend to a finite limit as $\kappa \to \kappa_0$, and A (hence D) will have a pole at $\kappa = \kappa_0$.

With this modification, we can use the dipole representation to obtain solutions of the Helmholtz equation even for $\kappa = \kappa_0$. For this purpose, we begin by writing

$$\psi(\mathbf{r}) = \int_0^L K(\mathbf{r}, s')\, D(s')\, ds',$$

where the use of $\mathbf{r}$ in the kernel rather than s is meant to indicate that $\mathbf{r}$ is an arbitrary point inside V. Using Eq. (25), we can write:

$$\psi(\mathbf{r}) = A \int_0^L [K(\mathbf{r}, s') - K_0(\mathbf{r}, s')]\, D_0(s')\, ds' \\ + A \int_0^L K_0(\mathbf{r}, s')\, D_0(s')\, ds' + \int_0^L K(\mathbf{r}, s')\, D_1(s')\, ds', \qquad (29)$$

where only the second term on the right exhibits a singularity as $\kappa \to \kappa_0$. This term is A times the eigenfunction of the homogeneous equation, and we will now show that this function is zero in the region of interest.

For the internal Dirichlet problem, for example, the partner problem is the external Neumann problem, and if κ_0 is an eigenvalue for the latter, the normal derivative of the external eigensolution, ψ_n', will vanish on the boundary. As a result of the identification of $D(\mathbf{r}) \equiv \psi(\mathbf{r}) + \psi'(\mathbf{r})$ following Eq. (23), this implies that $\psi_n(\mathbf{r})$ also vanishes on the boundary. Further, since D satisfies the homogeneous equation, $\psi(\mathbf{r})$ vanishes on the boundary as well and therefore $\psi(\mathbf{r})$ vanishes throughout the interior region. Similar arguments lead to the conclusion that for either an internal or external region or a Dirichlet or a Neumann problem, the eigensolution of the partner problem will vanish in the region of interest. [On the other hand, it is not true that an eigensolution of the problem of interest will vanish outside the region of interest.]

Thus, even though $D(s)$ has a singularity due to the existence of a partner eigensolution, $\psi(\mathbf{r})$ is finite and the difficulty in the behavior of $D(s)$ is easily removed by dropping the second term when calculating $\psi(\mathbf{r})$.

We can obtain a solution to the Helmholtz equation in the limit $\kappa \to \kappa_0$ if we set $A(\kappa - \kappa_0) \equiv A_1$, and then set $\kappa = \kappa_0$ in the modified kernel, k. We then must solve

$$\frac{D_1(s)}{2} + \int_0^L k_0(s, s')\, D_1(s')\, ds' = \Phi(s) - A_1 \int_0^L \left(\frac{\partial K(s, s')}{\partial \kappa}\right)_{\kappa=\kappa_0} D_0(s')\, ds'.$$

This equation can be solved for D_1 since the kernel $k(s, s')$ will not generally have an eigenvalue at $\kappa = \kappa_0$. (Only if the eigenvalue were degerate would a second eigenvalue occur, and in such a case the technique described here could be generalized to remove the second eigenfunction as well.) Finally, the solution of the Helmholtz equation in the limit $\kappa \to \kappa_0$ is given by the limit of Eq. (29):

$$\psi(\mathbf{r}) = A_1 \int_0^L \left(\frac{\partial K(\mathbf{r}, s')}{\partial \kappa}\right)_{\kappa=\kappa_0} D_0(s')\, ds' + \int_0^L K_0(\mathbf{r}, s')\, D_1(s')\, ds',$$

which clearly produces a finite $\psi(\mathbf{r})$.

The techniques in this paper have been used to obtain accurate solutions of the interior Dirichlet problem. For this interior problem, the kernel was constructed using the Neumann function N_1 instead of $-iH_1^{(1)}$ to obtain an integral equation with only real variables. This kernel led to a partner problem whose eigenvalues associated with the external Neumann problem were successfully removed as above. If the Hankel function had been used instead such eigenvalues would not have occurred but one would have twice as many variables to store in the computer since the integral equation is then complex.

In addition to partner eigenvalues, the interior problem also has true resonant solutions which produce very large solutions. These can be treated as above, except that the contribution to $\psi(\mathbf{r})$ from the term $A \int K_0(\mathbf{r}, s') D_0(s')\, ds'$ will no longer vanish and in fact will have a pole at the eigenvalue. In this case there seems to be little computational advantage to be gained by the separation of D into two parts. On the other hand, one can also visualize using the dipole representation in cases in which *physical* resonance of the system is not associated with the eigenvalues of the integral equation for a subregion, and so A cannot in fact have a pole at $\kappa = \kappa_0$. For example, the complete region of the problem may be divided into subregions of which the eigenvalues have little connection with the response of the system as a whole. If this situation can be identified a priori, it is then possible to modify the above procedure for the solution of the boundary value problem to improve numerical results. If, then, A does not have a pole as $\kappa \to \kappa_0$, it is clear from Eq. (27) and the fact that $D_1^{(0)} \to 0$ as $\kappa \to \kappa_0$ that

$$\lim_{\kappa \to \kappa_0} \int_0^L g(s)\, D_1^{(\Phi)}(s)\, ds = 0.$$

Hence, if $\kappa \approx \kappa_0$ the value of A is determined as the ratio of two integrals, each of which is almost zero. Thus errors in the D_1's are much increased in the evaluation of A, and in numerical calculations it is found that by far the largest contribution to the error in $\psi(\mathbf{r})$ comes from the error in A. To avoid this error, we may express

$$D_1^{(\Phi)}(s) \equiv \Delta(s) + \delta(s), \tag{30}$$

where

$$\frac{\Delta(s)}{2} + \int_0^L k_0(s, s')\, \Delta(s')\, ds' = \Phi_0(s)$$

and therefore

$$\frac{\delta(s)}{2} + \int_0^L k(s, s')\, \delta(s')\, ds' = \Phi(s) - \Phi_0(s) - \int_0^L [k(s, s') - k_0(s, s')]\, \Delta(s')\, ds'.$$

In these relations, k_0 is the kernel k and Φ_0 is the value of Φ in the limit $\kappa \to \kappa_0$. Then, since

$$\int_0^L g(s)\, \Delta(s)\, ds = 0,$$

$$A = - \int_0^L g(s)\, \delta(s)\, ds \Big/ \int_0^L g(s)\, D_1^{(0)}(s)\, ds.$$

This expression for A is a significant improvement over Eq. (27) because both $\delta(s)$ and $D_1^{(0)}(s)$ vanish in the limit $\kappa \to \kappa_0$ as a result of the fact that the corresponding right-hand side of the integral equation vanishes. This is to be contrasted with the [illegible] $D_1^{(0)}$ does not. In the new formulation, errors in A are indeed generated by errors in the right-hand side of the integral equation, but these are much more controllable than are errors in the *solution* of the integral equation. The significant feature of this approach is that errors in $\Delta(s)$ do not contribute to errors in A. Greatly improved accuracy in numerical calculations has been achieved using this formulation.

In this section we have shown how the dipole treatment can be modified to eliminate difficulties normally encountered at the eigenvalues of a partner problem. It is of course clear that if more than one such eigenvalue occurs near the chosen value of κ, the technique can easily be generalized to remove any desired set of eigenfunctions. Since the removal technique given here is inherently very simple, there is no compelling argument based on the eigenvalue difficulty for using the somewhat more complicated Helmholtz representation as suggested by Schenck [19], Burton and Miller [22] and Kleinman and Roach [3]. Using that representation, the lack of uniqueness can be removed in a general way by the introduction of a second integral relation which is normally redundant but becomes essential at the eigenvalue. One must then effectively deal with two kernels, the "charge" and the "dipole" Green's functions of the Helmholtz representation, and this seems to add substantially to the complexity of the numerical calculations. In addition, the inhomogeneous term in the integral equation involves the integral of a kernel function over the boundary value of the solution, thus creating further complexity and producing another source of numerical error in the calculations. In the approach developed here, it is necessary to make an a priori determination of the unwanted eigenvalues and their eigenfunctions, but once that is done these results can be stored for future use.

CONCLUSION

In this paper we have developed two ideas, both of which are of value for either improving the accuracy or extending the range of applicability of the dipole representation to the solution of the Helmholtz equation.

First, we provide an analysis of the properties of the dipole integral equation to obtain the form of the solution near a corner on the boundary. This analysis depends on the singular nature of the integral equation, and provides an alternative development to those based on the Helmholtz partial differential equation, although the results are different from but consistent with the latter. Secondly, we show how the dipole representation can be easily modified in many cases to remove a fundamental obstacle to its direct application.

In a subsequent paper we show that in fact the dipole representation is capable of excellent numerical accuracy for a variety of problems.

ACKNOWLEDGMENT

The author would like to thank Dr. Joseph V. Lepore of Lawrence Berkeley Laboratory for extensive discussions regarding many parts of this paper. He also wishes to thank Professor Bruce Bolt, Director of the University of California Seismology Laboratory for stimulating an interest in this problem. Finally, thanks are also due to Drs. Paul Concus, Loren Meissner, and Grove Nooney of the Lawrence Berkeley Laboratory for numerous discussions on various topics in relation to this paper.

REFERENCES

1. "Sur un nouvelle méthode pour la résolution du problème de Dirichlet," Ofr. Forh. Akad., Stockholm, 1900.
2. See, for example, H. LAMB, "Hydrodynamics," 6th ed., Art. 290, Dover, New York, 1945.
3. R. E. KLEINMAN AND G. F. ROACH, *SIAM Rev.* **16** (1974), 214.
4. See, for example, R. COURANT AND D. HILBERT, "Methods of Mathematical Physics," Interscience, New York, 1953. p. 159 ff.
5. M. A. JASWON, *Proc. Roy. Soc. London Ser. A.* **275** (1963), 23.
6. G. T. SYMM, *Proc. Roy. Soc. London Ser. A.* **275** (1963), 33.
7. K. K. MEI, "Scattering of Radio Waves by Rectangular Cylinders," Ph.D. dissertation, University of Wisconsin, 1962.
8. K. K. MEI AND J. G. VAN BLADEL, *IEEE Trans. Antennas and Propagation* **AP-11** (1963), 185.
9. L. G. COPLEY, *J. Acoust. Soc. Amer.* **44** (1968), 41.
10. G. J. BURKE, E. K. MILLER, A. J. POGGIO, G. M. PJERROU, B. J. MAXUM, AND W. MEECHAM, *J. Computational Phys.* **10** (1972), 22.
11. E. K. MILLER, A. J. POGGIO, G. J. BURKE, AND SELDEN, *Canad. J. Phys.* **50** (1972), 879.
12. E. K. MILLER, A. J. POGGIO, AND G. J. BURKE, *J. Computational Phys.* **12** (1973), 24, 210.
13. J. MEIXNER, *Ann. Physics* **6** (1949), 1.
14. A. W. MAUE, *Z. Physik.* **126** (1949), 601.
15. F. OBERHETTINGER, *Commun. Proc. Appl. Math.* **7** (1954), 551.
16. H. M. NUSSENZVEIG, *Philos. Trans. Roy. Soc. London Ser. A* **252** (1959), 1.
17. N. M. WIGLEY, *J. Math. Mech.* **13** (1964), 549.
18. L. G. COPLEY, *J. Acoust. Soc. Amer.* **44** (1968), 28.
19. H. A. SCHENCK. *J. Acoust. Soc. Amer.* **44** (1968), 41.
20. W. POGORZELSKI, "Integral Equations and Their Applications," Pergamon Press, Long Island City, N. Y., 1966, Chap. II.
21. G. CHERTOCK, *J. Acoust. Soc. Amer.* **47** (1970), 387.
22. A. J. BURTON AND G. F. MILLER, *Proc. Roy. Soc. London Ser. A* **323** (1971), 201.
23. See, for example, P. M. MORSE AND H. FESHBACH, "Methods of Theoretical Physics," McGraw-Hill, New York, 1953, Eq. (7.2.18). (Note that their Green's function satisfies $(\nabla^2 + k^2)G(\mathbf{r}) = -4\pi\delta(\mathbf{r})$.)
24. Reference [20, p. 231].
25. I. S. GRADSHTEYN AND I. M. RYZHIK, "Tables of Integrals, Series, and Products," Academic Press, New York, 1965, p. 284, Eq. (3.194.1).
26. A. ERDÉLYI *et al.*, *Higner Transcendental Functions*, McGraw-Hill, New York, 1953, Vol. 1, Eqs. (2.9.34), (2.9.9), (2.9.13), and (1.2.6).
27. Reference [20, Chap. XVII].
28. Reference [20, Chap. XVI].
29. Reference [20, pp. 516–517].
30. Reference [4, p. 155].